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SUMMARY

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The stability of spherical plate elements in both elastic and fully plastic ranges, is treated. Such elements fall in the transition region between axially loaded circular flat plates, where boundary conditions are significant, and spherical shells, where boundary conditions are insignificant. In this transition region a linear stability theory is used to investigate the effect of various boundary conditions and also to determine, in some detail, the regions in which the axisymmetric and asymmetric buckling modes govern.

AUTHOR

SYMBOLS

A_{ij}	=	plasticity coefficient matrix
$\bar{A}$	=	plasticity parameter $= (\frac{1}{2})A_{12}/A_{11}$
B	=	axial rigidity: elastic $Et/(1-\nu_e^2)$; plastic $= E_s t/(1-\nu_p^2)$
D	=	flexural rigidity: elastic $= Et^3/12(1-\nu_e^2)$; plastic $= E_s t^3/12(1-\nu_p^2)$
d	=	diameter of the base circle (dimensionless)
E_s	=	secant modulus
E_t	=	tangent modulus
E	=	elastic modulus
F	=	stress function
k	=	buckling coefficient $= \sigma t d^2/\pi^2 D$
m	=	number of modal circles
M	=	bending moment resultant per unit length
n	=	number of nodal lines
N	=	direct stress resultant per unit length
p	=	external pressure
r	=	radial coordinate (dimensionless)
R	=	radius of the sphere of which the cap is a part (dimensionless)
t	=	thickness of the shell (dimensionless)
u_ϕ, u_θ, w	=	dimensionless displacements
Z	=	dimensionless shell curvature parameter $= (d^2/Rt)(1-\nu_p^2)^{\frac{1}{2}}$
α	=	dimensionless buckling stress : elastic $= (\sigma t/D)^{\frac{1}{2}}$; plastic $= (\sigma t/DA_{11})^{\frac{1}{2}}$

β	=	dimensionless shell geometry parameter: elastic = $(Et/DR^2)^{\frac{1}{2}}$, plastic = $[E_t t/D{A_{11}}^2 R^2]^{\frac{1}{2}}$
ϵ	=	direct strain variation
$\bar{\eta}$	=	plasticity reduction factor
ν_e	=	elastic Poisson ratio
ν_p	=	plastic Poisson ratio = $\frac{1}{2}$
σ	=	constant compressive stress at buckling = $pR/2t$
χ	=	curvature variation
∇^2	=	Laplacian operator, $\partial^2/\partial r^2 + (1/r)\,\partial/\partial r + (1/r^2)\partial^2/\partial\theta^2$

STABILITY OF SPHERICAL PLATES

Introduction

The use of spherical plates as pressure vessel closures in aerospace vehicles has generated renewed interest in the stability of spherical plates under external pressure. The term spherical plates refers to those spherical elements that fall in the transition range between axially loaded circular flat plates where boundary conditions are significant and spherical shells where boundary conditions are insignificant.

In this paper, elastic and plastic stabilities of spherical plates are studied using small deflection theory to investigate the influence of various boundary conditions and also to determine in some detail the regions in which the axisymmetric and asymmetric buckling modes govern. While linear stability theory is known to yield buckling loads higher than experimental results for spherical shells, it is also recognized that linear stability theory is in agreement with experiments on flat plates. As a consequence, the use of linear theory for spherical plates may provide some useful results.

When a spherical shell buckles under external pressure, the buckle wavelength is confined to a small portion of the surface and the critical stress can be evaluated without specific reference to the boundary conditions. As another limiting case, let us consider the instability of a flat circular plate under axial compressive loading. Here the half wavelength of the buckling mode is of the same order of magntide as the plate diameter and the buckling stress is considerably influenced by the edge support.

It is evident from these two examples that the transition from the flat plate case to that of the full sphere can be effected through a series of spherical plates or shallow caps. Such a spherical plate is defined in the usual manner of shallow shell theory: its vertical rise is small compared to a characteristic horizontal length, the buckle half wavelength or the diameter of base circle.

The spherical plate solution will readily yield the results of the circular plate case by letting the appropriate term containing the height of the shell vanish. It is also possible to obtain the solution of the full sphere as a singular case of the same equations.

Governing Equations

In deriving the governing equations for the stability of spherical plates, the assumptions of shallow shell theory lead to simplifications in both the equilibrium and strain-displacement relationships. In the following, the distance from the axis of the shell to the edge of the cap is taken as the unit of length and hence all lengths are essentially dimensionless.

The strain displacement relationships are:

$$\epsilon_\theta = u_\phi/r + (1/r)(\partial u_\theta/\partial\theta) + w/R \qquad \chi_\theta = (1/r)(\partial w/\partial r) + (1/r^2)(\partial^2 w/\partial\theta^2) \quad (1)$$

$$\epsilon_\phi = \partial u_\phi/\partial r + w/R \qquad \chi_\phi = \partial^2 w/\partial r^2 \quad (2)$$

$$\epsilon_{\theta\phi} = 1/2\left[(1/r)(\partial u_\phi/\partial\theta) + r\,(\partial/\partial r)(u_\theta/r)\right] \quad \chi_{\theta\phi} = (\partial/\partial r)(1/r\,\partial w/\partial\theta) \quad (3)$$

In the above equations, ϵ, χ denote the direct strain and curvature variations; u_ϕ, u_θ and w the displacements in the meridional, circumferential and normal directions respectively. The direct strain variations ϵ_ϕ, ϵ_θ, $\epsilon_{\theta\phi}$, are seen to satisfy a compatibility relationship:

$$(1/r^2)\,\partial^2\epsilon_\phi/\partial\theta^2 - (1/r)\,\partial\epsilon_\phi/\partial r + (2/r)\,\partial\epsilon_\theta/\partial r$$
$$+ \partial^2\epsilon_\theta/\partial r^2 - (1/r^2)(\partial^2/\partial r\,\partial\theta)(r^2\epsilon_{\theta\phi}) = 1/r\ \nabla^2 w \quad (4)$$

The equilibrium equations for the buckling problem of a shallow spherical cap under external pressure, consistent with the strain displacement relations Eqs. (1) through (3), are

$$\partial(rN_\phi)/\partial r + \partial N_{\phi\theta}/\partial\theta - N_\theta = 0 \quad (5)$$

$$\partial(rN_{\phi\theta})/\partial r + \partial N_\theta/\partial\theta + N_{\phi\theta} = 0 \quad (6)$$

$$(1/r)\left[\partial^2(rM_\phi)/\partial r^2 + 2\,\partial^2(M_{\theta\phi})/\partial r\,\partial\theta - \partial M_\theta/\partial r + (2/r)\,\partial M_{\theta\phi}/\partial\theta\right]$$
$$+ (1/r^2)\,\partial^2 M_{\theta\phi}/\partial\theta^2 + 1/R\,(N_\theta + N_\phi) + p + \sigma t\ \nabla^2 w = 0 \quad (7)$$

In Eqs. (5) to (7), N, M refer to the direct stress and moment resultants respectively; p is the external pressure, σ the constant compressive stress (=pR/2t) at buckling, R the radius of the sphere of which the cap is a part, t the thickness of the shell and ∇^2 the Laplacian operator.

From Eqs. (5) and (6) it is readily seen that a stress function F can be introduced such that they are satisfied identically. The direct stress resultants then are derivable from the stress function F as shown:

$$
\begin{aligned}
N_\phi &= (1/r)\,\partial F/\partial r + (1/r^2)\,\partial^2 F/\partial\theta^2 \\
N_\theta &= \partial^2 F/\partial r^2 \\
N_{\phi\theta} &= -(\partial/\partial r)\,[(1/r)\partial F/\partial\theta\,]
\end{aligned}
\tag{8}
$$

The plastic stability theory used herein, following Ref. 1, is based on a deformation theory of plasticity. Hence the stress strain relationships for the spherical plate case are similar in form in both elastic and plastic ranges. It would be advantageous to derive the governing equations in terms of the plastic coefficients so that the elastic results are readily obtained by modifying the coefficients suitably.

The stress resultant-strain relationships in the plastic range (Ref. 1) for the spherical plate are:

$$N_\phi = B\,A_{11}(\epsilon_\phi + \bar{A}\,\epsilon_\theta) \qquad M_\phi = D\,A_{11}(\chi_\phi + \bar{A}\,\chi_\theta) \tag{9}$$

$$N_\theta = B\,A_{11}(\epsilon_\theta + \bar{A}\,\epsilon_\phi) \qquad M_\theta = D\,A_{11}(\chi_\theta + \bar{A}\,\chi_\phi) \tag{10}$$

$$N_{\theta\phi} = B\,A_{11}(1 - \bar{A})\,\epsilon_{\theta\phi} \qquad M_{\theta\phi} = D\,A_{11}(1 - \bar{A})\,\chi_{\theta\phi} \tag{11}$$

where B, D are the axial and flexural rigidities in the fully plastic range, given by $B = E_s t(1-\nu_p^2)$ and $D = E_s t^3/12(1-\nu_p^2)$, with ν_p being the full plastic value of Poisson ratio (=1/2) and E_s the secant modulus.

A_{11} is a component of the plasticity coefficient matrix A_{ij}, which for the spherical case has the following form, (Ref. 1):

$$
\begin{pmatrix}
(3E_t/E_s + 1)/4 & (3E_t/E_s - 1)/2 & 0 \\
(3E_t/E_s - 1)/2 & (3E_t/E_s + 1)/4 & 0 \\
0 & 0 & 1
\end{pmatrix}
\tag{12}
$$

where E_t is the tangent modulus. Finally $\bar{A}$ in Eqs. (9) to (11) is a plasticity parameter given by:

$$\bar{A} = (1/2)\, A_{12}/A_{11} = (3E_t/E_s - 1)/(3E_t/E_s + 1) \tag{13}$$

It is clear from Eqs. (9) to (11), with the definitions of the various coefficients, that the above relationships can be carried over to the elastic range with the following modifications: $E_t = E_s = E$, the elastic modulus; hence $A_{11} = 1$. B, D now refer to axial and flexural rigidities given by $Et(1-\nu_e^2)$ and $Et^3/12(1-\nu_e^2)$ respectively (ν_e being the elastic Poisson's ratio). $\bar{A}$ is to be replaced by ν_e the elastic Poisson ratio.

With foregoing modifications all the results that are obtained in the plastic case can be written readily for the elastic case.

Then the stability problem of the spherical plates reduces to the solving of Eqs. (4) and (7). These can be modified to yield a pair of coupled equations in F and w by making use of the stress-strain relationships Eqs. (9) to (11), and the strain displacement relationships Eqs. (1) to (3) and Eq. (8). The final form of these governing equations are:

$$\nabla^2 \nabla^2 F = (E_t\, t/A_{11} R)\ \nabla^2 w \tag{14}$$

$$D A_{11}\, \nabla^2 \nabla^2 w + (1/R)\, \nabla^2 F + p + \sigma t\, \nabla^2 w = 0 \tag{15}$$

Before we deal with Eqs. (14) and (15) as such, it is advantageous to study the flat circular plate and full sphere cases first.

Flat Circular Plate

For a flat circular plate under axial compression with $R \rightarrow \infty$ in Eqs. (14) and (15), we obtain the following governing equations:

$$\nabla^4 w + (\sigma t/DA_{11})\, \nabla^2 w + (p/DA_{11}) = 0 \tag{16}$$

If we let p = 0 in Eq. (16), we have

$$\nabla^2 (\nabla^2 + \alpha^2)\, w = 0 \tag{17}$$

where $\alpha^2 = (\sigma t/DA_{11})$. The general solution of Equation (17), with the requirement that w, $1/r\ \partial w/\partial r$ and $\partial^2/\partial r^2$ be finite at r = 0, can be written as

$$w = \sum_{n=0}^{\infty} \left[C_{0n} r^n + C_{1n} J_n(\alpha r) \right] \cos n\theta \tag{18}$$

when n represents the number of nodal lines on the deformed surface.

For the typical boundary conditions of simple support and full edge fixity at r = 1, we obtain the following characteristic equations for determining α :

Simply supported edge: $\quad w = 0, \quad M_r = 0$

Plastic: $\quad \alpha J_n(\alpha) - (1 - \bar{A})\, J_{n+1}(\alpha) = 0, \ \alpha = (\sigma t/DA_{11})^{\frac{1}{2}} \quad$ (19a)

Elastic: $\quad \alpha J_n(\alpha) - (1 - \nu_e) J_{n+1}(\alpha) = 0, \ \alpha = (\sigma t/D)^{\frac{1}{2}} \quad$ (19b)

Completely fixed edge: $\quad w = 0, \quad \partial w/\partial r = 0$

Plastic: $\quad J_{n+1}(\alpha) = 0, \quad \alpha = (\sigma t/DA_{11})^{\frac{1}{2}} \quad$ (20a)

Elastic: $\quad J_{n+1}(\alpha) = 0 \quad \alpha = (\sigma t/D)^{\frac{1}{2}} \quad$ (20b)

For a given n, the roots of Eqs. (19) and (20) correspond to increasing number of nodal circles. Thus the first (lowest root) of Eq. (19) for n = 0 would correspond to the axisymmetric case with one nodal circle at the edge. If m denotes the number of nodal circles, then we can characterize the solutions of Eqs. (19) and (20) by their m, n number, Fig. (1) shows some typical shapes of the modes taken from Ref. (2).

The following table shows some of the typical values, for the first four modes, of an elastic buckling coefficient $k = \sigma t\, d^2/\pi^2 D = \alpha^2 d^2/\pi^2$, where d is the diameter of the plate. Since all linear dimensions are normalized with respect to the base radius, k becomes equal to $4\,\alpha^2/\pi^2$. In Eq. (19b) ν_e has been taken equal to 0.3.

TABLE 1. ELASTIC BUCKLING COEFFICIENTS FOR FLAT CIRCULAR PLATE

	Governing Modes			
Edge Conditions	m = 1 n = 0	m = 1 n = 1	m = 1 n = 2	m = 2 n = 0
Simply Supported	1.70	5.32	10.05	11.78
Fully Fixed	5.95	11.3	16.5	20.0

In order to separate the plasticity effects it is useful to define a plasticity reduction factor $\bar{\eta}$, following Ref. (1), given by

$$\bar{\eta} = (\sigma/D)_{\text{plastic}} / (\sigma/D)_{\text{elastic}}$$

In Table 2 below, we have the results of Eq. (19a) with n = 0 given in terms of $\bar{\eta}$ for different E_t/E_s ratios.

TABLE 2. PLASTICITY REDUCTION FACTORS FOR A SIMPLY SUPPORTED FLAT CIRCULAR PLATE

E_t/E_s	$\bar{\eta}$
1.0	1.0
0.75	0.765
0.50	0.526
0.25	0.278
0	0

Full Sphere

Equations (14) and (15) are transformed, after operating with ∇^2, into the following:

$$\nabla^6 w + (\sigma t/DA_{11}) \nabla^4 w + (E_t t/A_{11}^2 DR^2) \nabla^2 w = 0 \tag{22}$$

$$\nabla^6 F + (\sigma t/DA_{11}) \nabla^4 F + (E_t t/A_{11}^2 DR^2) \nabla^2 F = -(E_t t/DA_{11}^2 R)p \tag{23}$$

We assume a constant stress state throughout the region under pressure given by $\nabla^2 F = - pR$ so that Equation (22) is satisfied identically.

Then a suitable form for displacement w would be

$$w = C_n J_n (kr) \cos n\theta \tag{24}$$

By substituting Eq. (24) into Eq. (22) and obtaining a minimum condition for ($\sigma t/DA$) we find that

$$(\sigma t/DA_{11})_{min} = 2(E_t t/DA_{11}^2 R^2)^{\frac{1}{2}} \tag{25}$$

or

$$\sigma_{cr} = 3 (1 - \nu_e^2)^{-\frac{1}{2}} \eta (Et/R) \tag{26}$$

where

$$\eta = (1 - \nu_e^2) / (1 - \nu^2)^{\frac{1}{2}} (E_s/E) (E_t/E_s)^{\frac{1}{2}} \tag{27}$$

In Eq. (27) ν is the current Poisson ratio appearing in $D = E_s t^3 / 12(1-\nu^2)$ as in Ref. 1. It is readily seen that by taking $\eta = 1$ in Eq. (26) we get the elastic critical stress.

It is clear that the expression for σ_{cr} in Eq. (26) is independent of n in the expression for w in Eq. (24). Hence for both axi- and asymmetric modes, we get the same critical stress for the sphere.

Spherical Plate

In the case of a spherical plate, where both curvature and boundary affect the buckling stress, a simple assumption on the stress state cannot be made as in the case of the sphere. Hence, we must consider the general solutions of Eq. (22) and (23) which can be written conveniently as

$$\nabla^2 (\nabla^2 + k_1^2)(\nabla^2 + k_2^2)\, w = 0 \tag{28}$$

$$\nabla^2(\nabla^2 + k_1^2)(\nabla^2 + k_2^2)\, F = - k_1^2\, k_2^2\, Rp \tag{29}$$

where

$$k_1^2,\ k_2^2 = \alpha^2/2 \pm \left[(\alpha^2/2)^2 - \beta^2\right]^{\frac{1}{2}} \tag{30}$$

and

$$\alpha^2 = \sigma t/DA_{11},\ \beta^2 = E_t t/DA_{11}^2\, R^2 \tag{31}$$

It is seen from Eq. (25) that the case of full sphere corresponds to $\alpha^2/2 = \beta$ or $k_1 = k_2$ in Eq. (30). Hence for the spherical plate we need consider only $k_1 \neq k_2$.

The solutions of Eq. (28) and (29) can now be written as:

$$w = \sum_{n=0}^{\infty} C_{on} r^n + C_{1n} J_n (k_1 r) + C_{2n} J_n (k_2 r) \quad \cos n\theta \tag{32}$$

$$F = - pR/4r^2 - (E_t t/A_{11} R) \sum_{n=0}^{\infty} A_{on} r^n + (C_{1n}/k_1^2) J_n(k_1 r) + (C_{2n}/k_2^2) J_n(k_2 r) \cos n\theta \tag{33}$$

Where the finiteness of w, $(1/r)\partial w/\partial r$, $\partial^2 w/\partial r^2$, F, $1/r\, \partial F/\partial r$ and $\partial^2 F/\partial r^2$ at $r = 0$ is taken into account.

While it is possible to treat the problem in its entirety, it is advantageous to consider the axisymmetric case first.

Axisymmetric Case

The solution for the axisymmetric case (n = 0) is:

$$w = C_{00} + C_{10} J_0 (k_1 r) + C_{20} J_0 (k_2 r) \tag{34}$$

$$F = pR(r^2/4) - (E_t t/A_{11} R) \left[A_{00} + (C_{10}/k_1^2) J_0(k_1 r) + (C_{20}/k_2^2) J_0(k_2 r)\right] \tag{35}$$

In Eqs. (34) and (35) only three constants C_{00}, C_{10} and C_{20} are of importance, since any specification on the non-vanishing stress resultants N_θ and N_ϕ involves only the derivatives of F, thus not involving A_{00} at all.

The usual boundary conditions of simple support and complete edge fixity which normally give two conditions on w and its derivatives, are not enough to determine the three constants involved. Thus extra conditions involving the derivatives of F are required. The additional boundary conditions can be specifications on the membrane stress resultants N_θ , N_ϕ or the tangential displacement u_ϕ or its derivative, which can be written in terms of N_θ , N_ϕ through the use of Eqs. (9) and (10). For each of these groups of boundary conditions we can set up a system of algebraic equations leading to a characteristic equation whose eigenvalues give us the buckling coefficient corresponding to various modes.

The following conditions represent simple support types of boundary conditions at r = 1

$$w = 0, \quad M_\phi = 0, \quad N_\phi = -pR/2 \tag{36}$$

$$w = 0, \quad M_\phi = 0, \quad N_\phi = N_\theta = -pR/2 \tag{37}$$

$$w = 0, \quad M_\phi = 0, \quad u_\phi = -(pR/2)(1-\bar{A})A_{11}/E_t t \tag{38}$$

Of these conditions, Eqs. (36, 37) represent nonvanishing stress resultants, Eq. (38) the nonvanishing tangential displacement. The corresponding characteristic equations are:

$$[k_1^3 J_0(k_1) J_1(k_2) - k_2^3 J_0(k_2) J_1(k_1)] + (1-\bar{A})(k_2^2 - k_1^2) J_1(k_1) J_1(k_2) = 0 \quad (39)$$

$$(k_1^2 - k_2^2) J_0(k_1) J_0(k_2) + (1-\bar{A}) [k_1 J_0(k_2) J_1(k_1) - k_2 J_0(k_1) J_1(k_2)] = 0 \quad (40)$$

$$k_1 k_2 (k_1^2 - k_2^2) J_0(k_1) J_0(k_2) + k_1 k_2 (1-\bar{A}) [k_1 J_0(k_2) J_1(k_1) - k_2 J_0(k_1) J_1(k_2)]$$

$$+ (1+\bar{A}) [k_1 J_0(k_1) J_1(k_2) - k_2 J_0(k_2) J_1(k_1)] + (1-\bar{A}^2)(k_2^2 - k_1^2) J_1(k_1) J_1(k_2) = 0 \quad (41)$$

It is to be noted that the condition in Eq. (37) would imply that $N_\theta + N_\phi = -pR$ at the edge, that is, the value of $\nabla^2 F = -pR$ at the edge. This would in its turn, result in the vanishing of the constant C_{00} in Eq. (34) which is equivalent to considering a fourth order equation for w. Such a condition results in simpler expressions for the characteristic equations as seen from Eqs. (40) and (41).

In the completely fixed case, again, one may find an appropriate set of conditions that specify either membrane stress resultants or displacements. But we can simplify the expressions by assuming that $N_\theta = N_\phi = -pR/2$ or rather, $\nabla^2 F = -pR$ holds in this case also. Then we find $C_{00} = 0$ once again. Grouping both these conditions we may write the types and their resulting characteristic equations as:

Simply Supported Type: $w = 0, M_\phi = 0, \nabla^2 F = -pR$ at $r = 1$

$$(k_2^2 - k_1^2) J_0(k_1) J_0(k_2) + (1-\bar{A})[k_1 J_0(k_2) J_1(k_1) - k_2 J_0(k_1) J_1(k_2)] = 0 \quad (42)$$

Completely Fixed Type: $w = 0, dw/dr = 0, \nabla^2 F = -pR$

$$k_1 J_0(k_2) J_1(k_1) - k_2 J_0(k_1) J_1(k_2) = 0 \quad (43)$$

From Eqs. (27) through (31) we can find the roots of a characteristic equation for a given β, in accordance with a given set of boundary conditions.

<u>Asymmetric Case (n > 0)</u>

For n > 0, we find now that all the stress and strain components are no longer independent of θ, the circumferential coordinate and hence, we find that cross terms, like $N_{\theta\phi}$, $\epsilon_{\theta\phi}$, $\chi_{\theta\phi}$, $M_{\theta\phi}$, enter into the problem. Further the displacement in the circumferential direction, u_θ does not vanish in general.

Again from Eq. (33) it is seen that A_{on} cannot be ignored if $n > 0$. Thus with 4 constants to be determined, we need conditions on other stress resultants like $N_{\theta\phi}$, or the corresponding strains in order to set up the eigenvalue problem. However, if we assume as in the axisymmetric case, Eqs. (42) and (43), that $F = -pR$ is a condition that is always valid at the edge, then any requirement that w be zero at the boundary implies the vanishing of C_{on} for any n. Hence, the eigenvalue problem is simplified. Therefore, we can write the following characteristic equations for the different boundary conditions.

Simply Supported Type: $w = 0$, $M_\phi = 0$, $\nabla^2 F = -pR$ at $r = 1$

$$(k_2^2 - k_1^2) J_n(k_1) J_n(k_2) + (1 - \bar{A}) [k_1 J_n(k_2) J_{n+1}(k_1) - k_2 J_n(k_1) J_{n+1}(k_2)] = 0 \quad (44)$$

Completely Fixed Type: $w = 0$, $\partial w/\partial r = 0$; $\nabla^2 F = -pR$ at $r = 1$

$$k_1 J_n(k_2) J_{n+1}(k_1) - k_2 J_n(k_1) J_{n+1}(k_2) = 0 \quad (45)$$

Eqs. (44) and (45) are seen to be the obvious generalizations of corresponding axisymmetric cases of Eqs. (42) and (43). Thus we see for the types described above Eqs. (44) and (45) are valid for $n > 0$. Furthermore, from the symmetries of Bessel functions of integral orders, we see that $\pm\beta$ does not affect Eqs. (44) and (45). Hence only positive values of k_1, k_2 need be considered.

From Eqs. (32) and (33) which are the characteristic equations for the simple support and completely fixed types for all values of n, an infinite number of roots can be obtained for a given β. Each non-trivial root, for a given n corresponds to an increased number of nodal circles. Thus, the first root for any n, would give us a single nodal circle at the edge. If we once again denote the nodal circles by m, we can obtain from Eqs. (44) and (45), the eigen solutions (which corresponds to the buckling stress) for each value of β (which describes the geometry of the shell suitably) according to their modes (m, n specifications.)

Evidently, the solutions corresponding to the elastic case are given from Eqs. (44) and (45) by replacing $\bar{A}$ by ν_e and re-interpreting k_1 and k_2.

Thus for the elastic case, from Eq. (31) we have

$$\alpha^2 = \sigma t/D \text{ and } \beta^2 = Et/DR^2;\ k_1 k_2 = \beta \tag{46}$$

Numerical Results

Eqs. (44) and (45) have been solved for various values of β on a high speed digital computer for the elastic case with $E_t = E_s = E$; and ν_e the Poisson ratio being taken as $\nu_e = \nu_p = 1/2$ for ease of comparison with other E_t/E_s ratios. Eq. (44) corresponding to the simply supported type has then been solved for E_t/E_s ratios of 0.75, 0.50 and 0.25.

For convenience of comparison with the results from cylindrical shell studies, Ref. (3), the coordinates have been redefined. A buckling coefficient $k = \sigma t \partial^2/\pi^2 D$ where d is the base diameter, and a shell parameter Z given by, $Z = (d^2/Rt)(1-\nu_p{}^2)^{\frac{1}{2}} = (\sqrt{3}/2)(d^2/Rt)$ have been used; α and β are related to k and Z in the following manner

$$\begin{aligned} \text{Elastic:} \quad & k = \sigma t d^2/\pi^2 D = d^2\alpha^2/\pi^2 = 4\alpha^2/\pi^2 \\ \text{Plastic:} \quad & k = \sigma t d^2/\pi^2 D = (d^2/\pi^2)\, A_{11}\alpha^2 = 4\alpha^2 A_{11}/\pi^2 \end{aligned} \tag{47}$$

$$\begin{aligned} \text{Elastic:} \quad & Z = (\sqrt{3}/2)\, d^2/Rt = 2\sqrt{3}/Rt = (2/\sqrt{3})\, \beta \\ \text{Plastic:} \quad & Z = (\sqrt{3}/2)\, d^2/Rt = 2\sqrt{3}/Rt = (2/\sqrt{3})\, A_{11}\, (Et/E_s)^{-\frac{1}{2}} \beta \end{aligned} \tag{48}$$

In Eqs. (47) and (48), since all lengths are normalized with respect to the radius of the base circle, d = 2.

The case of the full sphere, that is, $\alpha^2 = 2\beta$, we have

$$\text{Elastic:} \quad k = (4\sqrt{3}/\pi^2)\, Z$$

$$\text{Plastic:} \quad k = (4\sqrt{3}/\pi^2)(E_t/E_s)^{\frac{1}{2}}\, Z$$

Figs. 2 and 3 show the k-Z plots for simply supported and clamped type edges for the elastic case. It is seen that various modes start from a flat plate value (Z = 0) and for large values of Z intersect each other, the intersections being more and more rapid as the higher modes are considered until a minimum curve drawn as shown in bold lines in Fig. 2 becomes indistinguishable from the straight line for these values of Z. This behavior is seen to be similar to that of cylindrical shells (Ref. 3). These minimum curves corresponding to buckling stress at a given Z value, are shown separately in Fig. 4 for the simply supported and the clamped type cases.

Returning to Fig. 2 we find that up to about Z = 9, the mode is the axisymmetric mode, while for higher values up to about Z = 30, the asymmetric modes prevail. For higher values axi- and asymmetric modes occur quite rapidly until the minimum curves coincides with the straight line which corresponds to the case of full sphere.

Fig. 5 presents the results of the elastic case in a slightly different form. The ordinate represents the critical stress normalized with respect to the critical stress of the spherical case and the abscissa represents $\lambda = \sqrt{\beta}$.

The k - Z plots of Fig. 6, shows the curves that correspond to various buckling modes indicated by m, n specifications, for E_t/E_s ratios of 1 and 0.25. It is a distinctive feature of the spherical plate problem that there is a single minimum line $k = (4\sqrt{3}/\pi^2)(E_t/E_s)^{\frac{1}{2}} Z$, for the value of $E_t/E_s \leq 1$, corresponding to both axi- and asymmetric modes as a contrasted with the cylindrical problem, Ref. 3 , where for the plastic case there are two minimum lines depending upon whether the governing mode is axi- or asymmetric.

To compare the plastic and elastic behaviors, using the plasticity reduction factor defined in Eq. (21), values of $\bar{\eta}$ have been determined for the different modes by using the data presented in Figure 6. These data are presented in Figs. 7 and 8 . For large Z values, where the straight line becomes the best approximation for the buckling stress as seen from Fig. 6 $\bar{\eta}$ is seen from Eq. (49) to be equal to (E_t/E_s) .

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3. Gerard, G., "Compressive Stability of Orthotropic Cylinders", Jour. of Aero. Sci. Vol. 29, No. 10, Oct. 1962, p. 1171-80

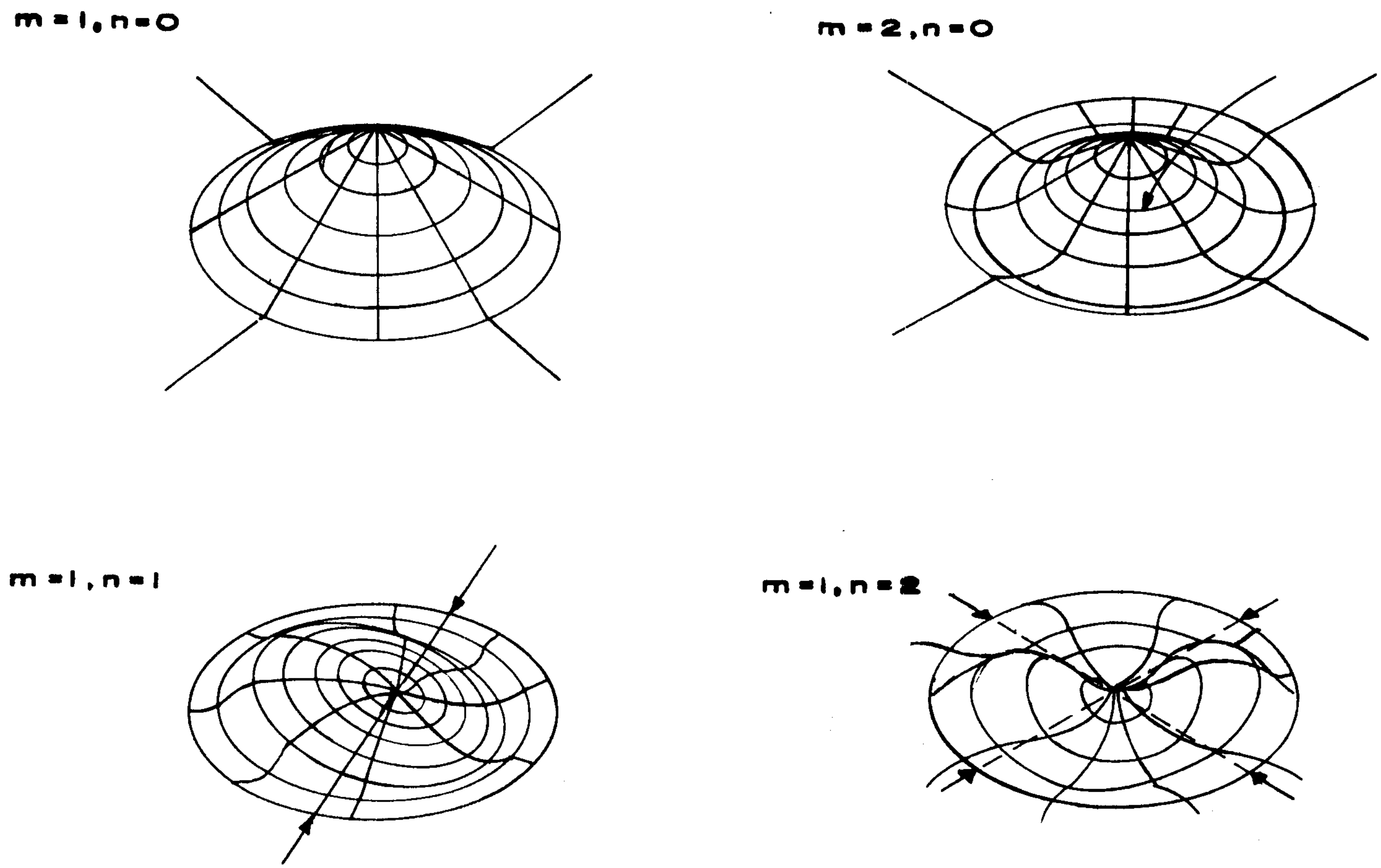


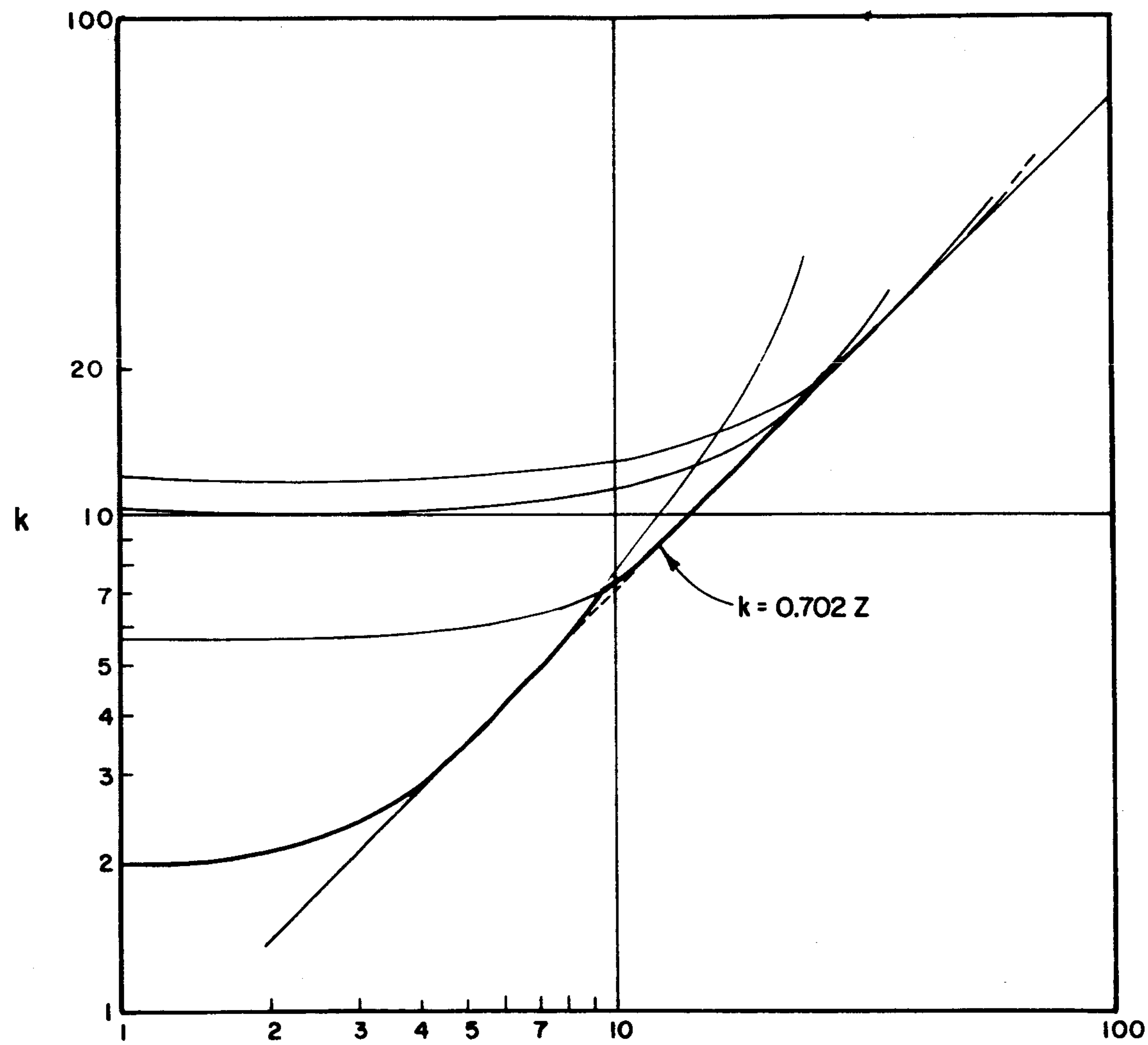
FIG. 1

SHAPES OF A FEW OF THE NORMAL MODES OF BUCKLING OF A CLAMPED CIRCULAR PLATE

FIG. 2

ELASTIC BUCKLING COEFFICIENT AS A FUNCTION OF Z

SIMPLY SUPPORTED TYPE



$Z = d^2/Rt(1-\nu_p^2)$

FIG. 3

ELASTIC BUCKLING COEFFICIENT AS A FUNCTION OF Z

CLAMPED TYPE

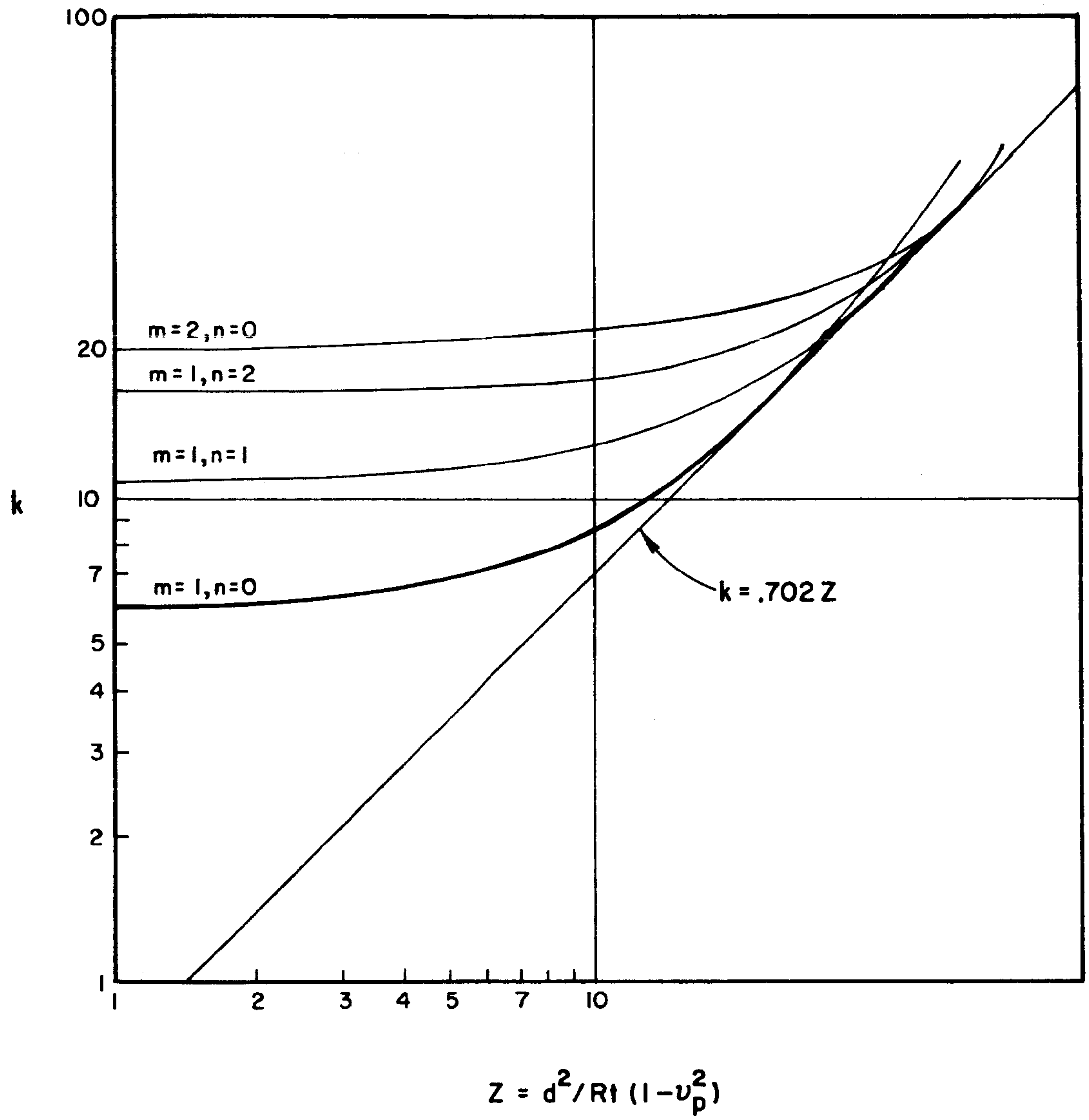
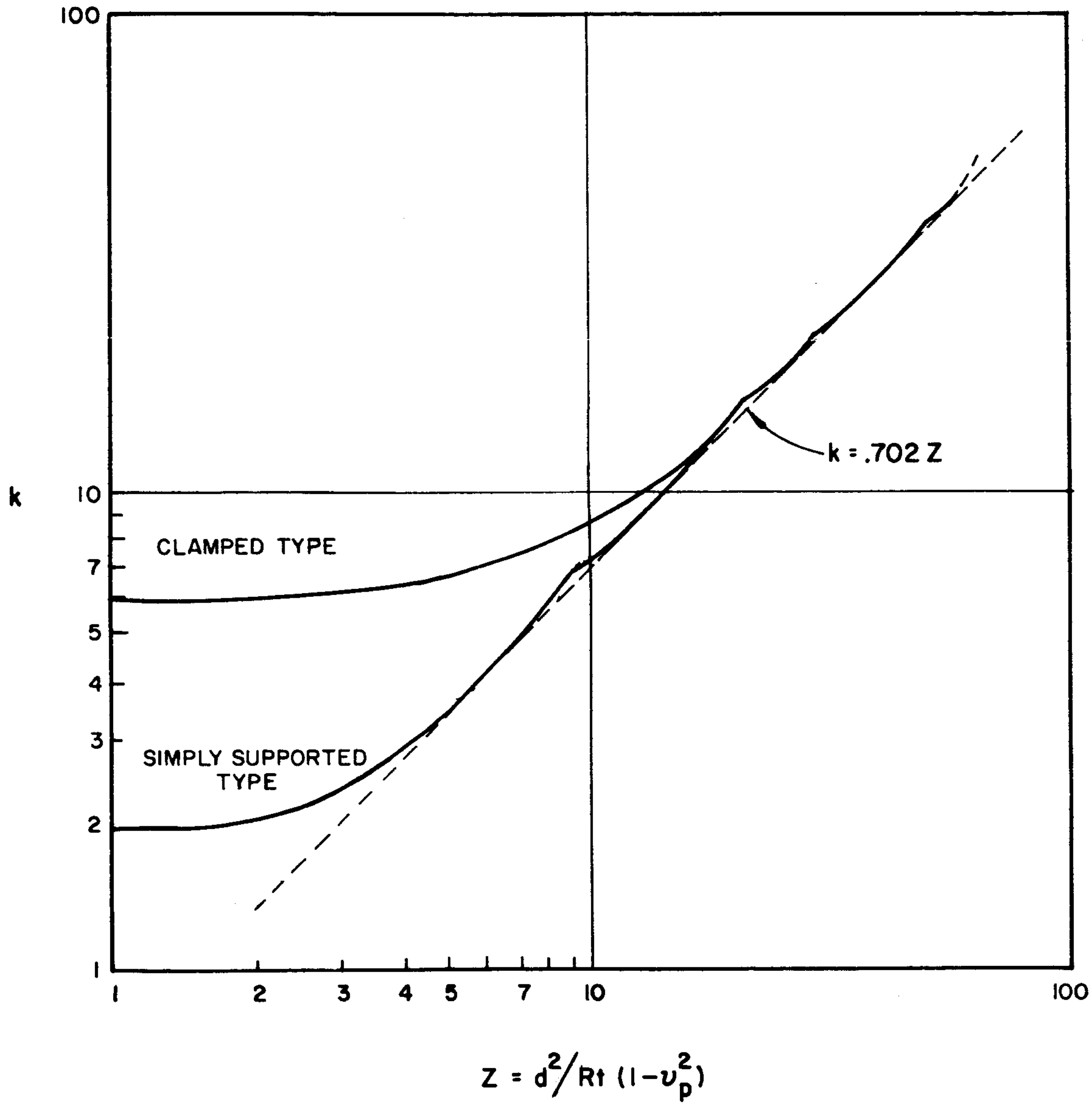


FIG. 4

ELASTIC BUCKLING COEFFICIENT AS A FUNCTION OF Z



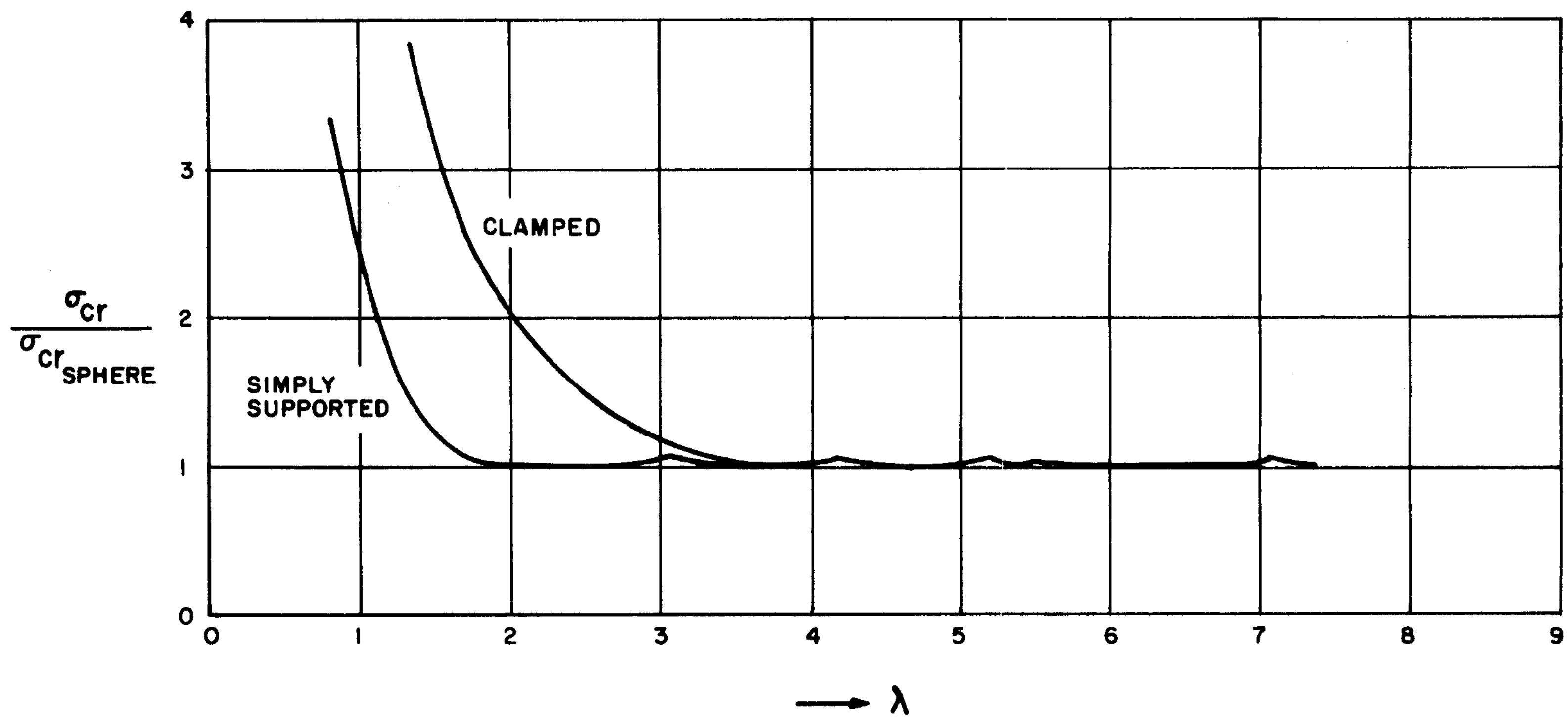


FIG. 5

CRITICAL PRESSURE AS A FUNCTION OF λ

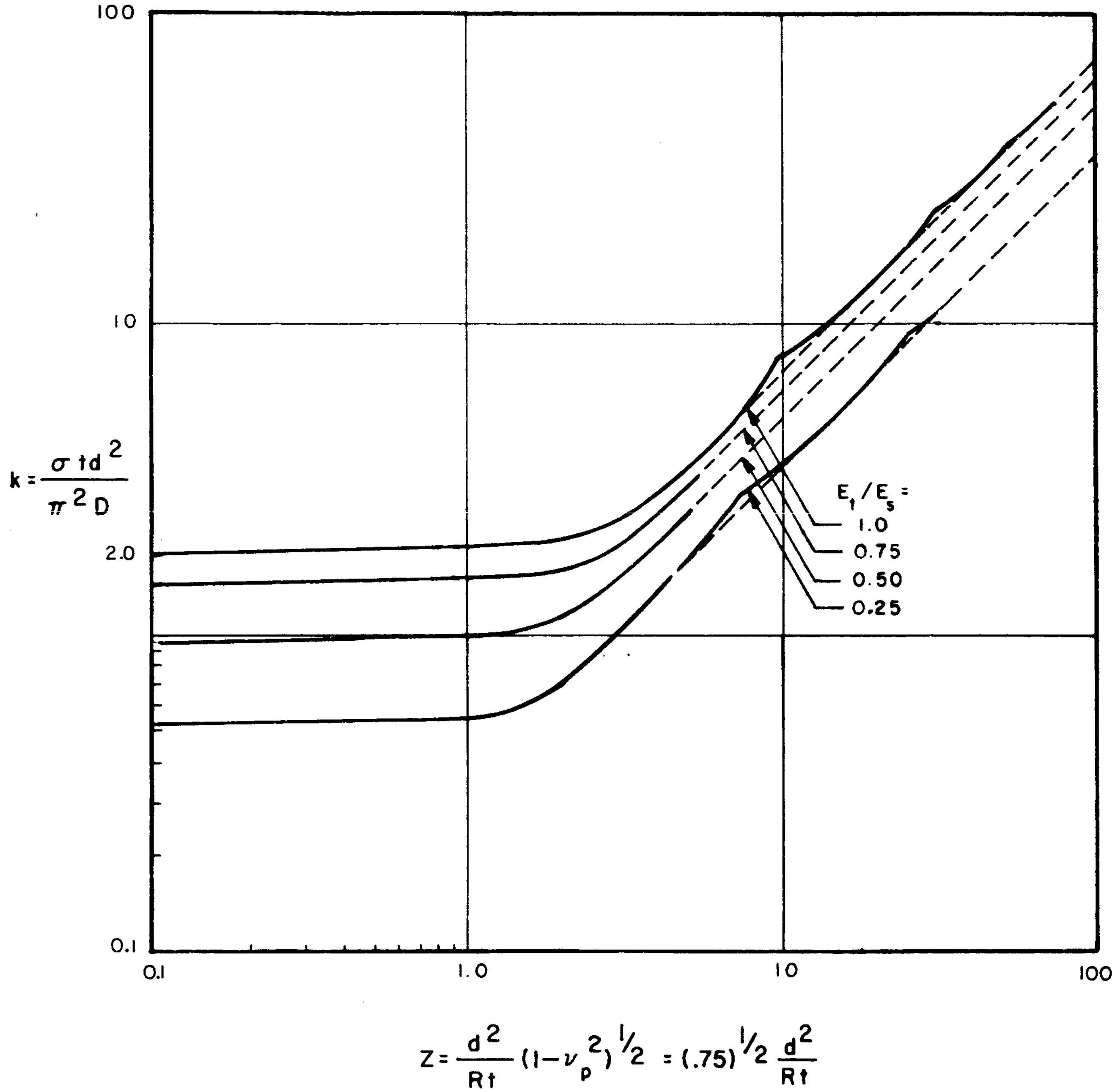


FIG.6

PLASTIC BUCKLING COEFFICIENT AS A FUNCTION OF Z

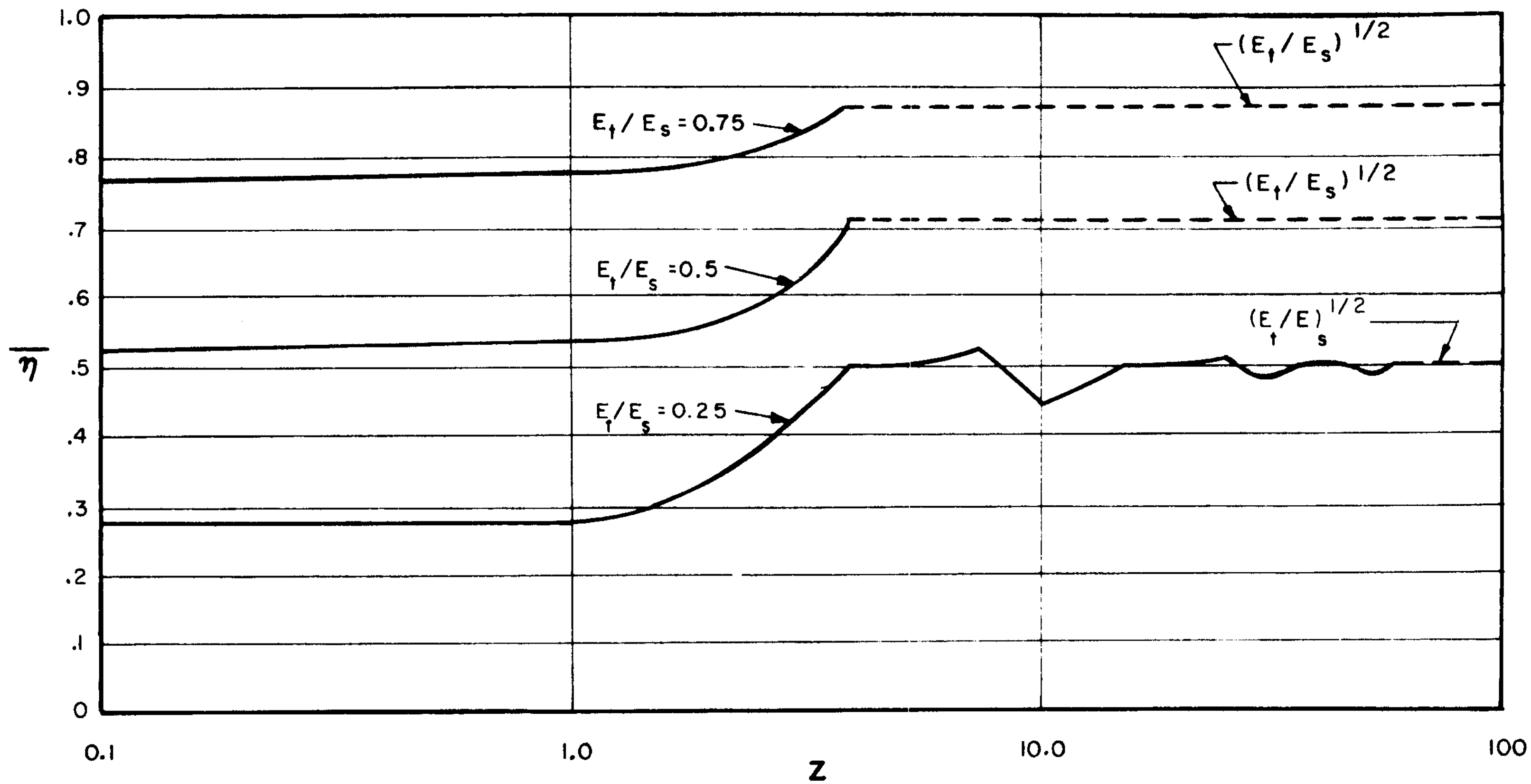


FIG. 7

PLASTICITY REDUCTION FACTOR AS A FUNCTION OF Z

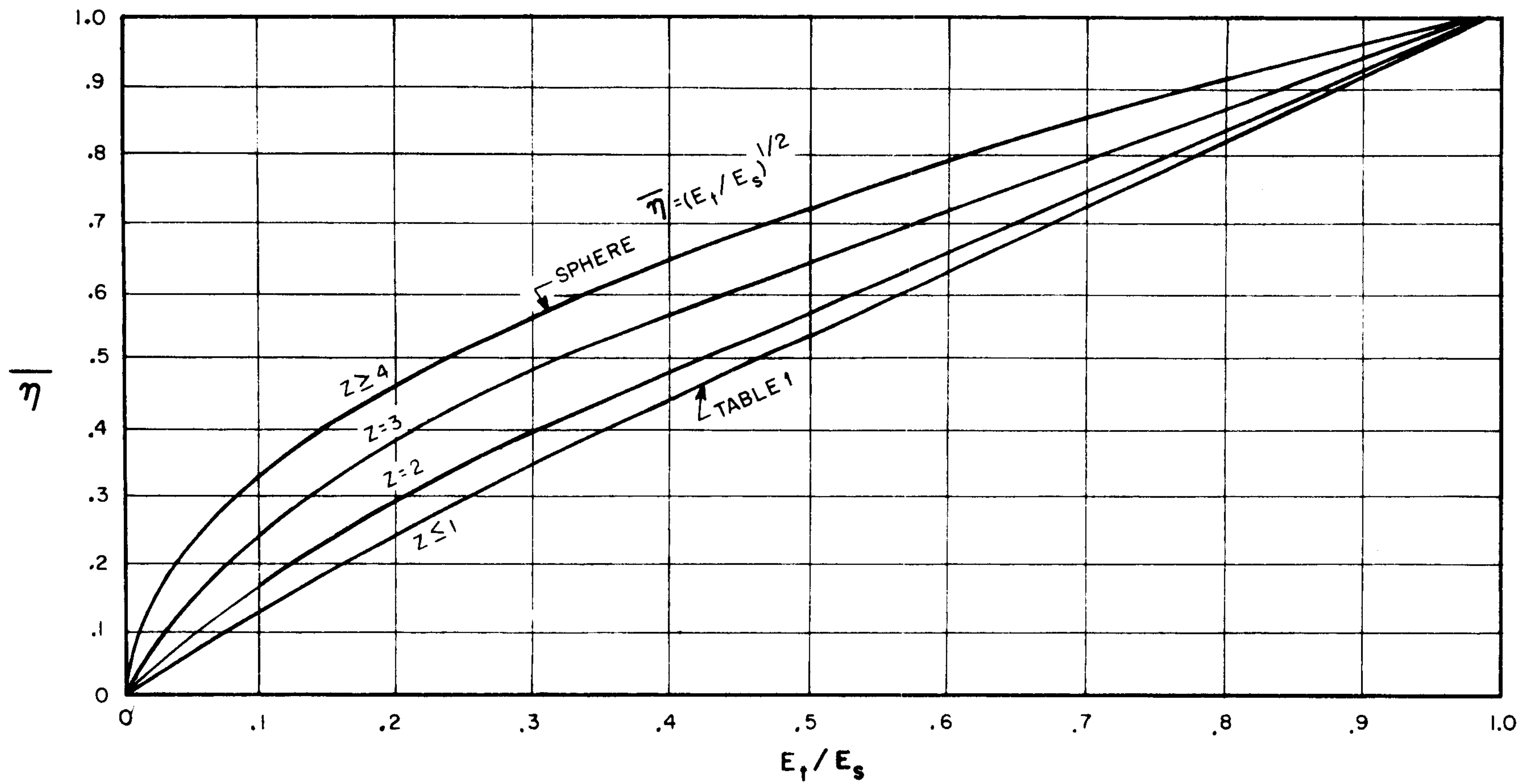


FIG. 8

PLASTICITY REDUCTION FACTOR AS A FUNCTION OF E_t/E_s